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THE BUCKLING ANALYSIS OF A ELASTICALLY CLAMPED RECTANGULAR PLATE

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RESEARCH ARTICLE

ABSTRACT: The paper analyses the stability of a rectangular plate which is elastically buckled along longitudinal edges and pressed by equally distributed forces. A general case is analysed – different stiffness elastic clamping and then special simpler cases are considered. Energy method is used in order to determine critical stress. Deflection function is introduced in a convenient way so that it reflects the actual state of the plate deformation in the best manner. In this way, critical stress is determined in analytic form suitable for analysis. With help of the equation it is easy to conclude how certain parameters influence the value of critical stress. The paper indicates how the obtained solution could be utilized for determining local buckling critical stress in considerably more complex systems – pressed thin-walled beams of an arbitrary length.

KEY WORDS: *buckling, rectangular plate, thin-walled beams*

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ANALIZA IZVIJANJA ELASTIČNO STEGNUTE PRAVOUGAONE PLOČE

REZIME: U radu je analizirana stabilnost pravougaone ploče koja je elastično izvijena duž uzdužnih ivica i pritisnuta podjednako raspoređenim silama. Analizira se opšti slučaj – različite krutosti tokom elastičnog ukljeŝtenja, a zatim se razmatraju posebni jednostavniji slučajevi. Energetski metod se koristi za određivanje kritičnog napona. Funkcija otklona je uvedena na odgovarajući način tako da na najbolji način odražava stvarno stanje deformacije ploče. Na ovaj način se kritični napon određuje u analitičkom obliku pogodnom za analizu. Uz pomoć jednačine lako je zaključiti kako pojedini parametri utiču na vrednost kritičnog napona. U radu je prikazano kako bi se dobijeno rešenje moglo koristiti za određivanje kritičnog napona lokalnog izvijanja u znatno složenijim sistemima – presovanim tankozidnim gredama proizvoljne dužine.

KLJUČNE REČI: *izvijanje, pravougaona ploča, tankozidne grede*

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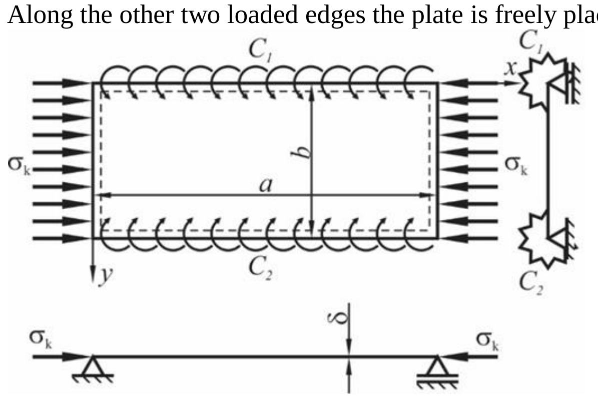
INTRODUCTION

In researching so called local buckling of thin-wall prismatic beams under a uniform compression load over the cross-section area, a problem occurs with determining critical stress, with the entire element as a whole as well as in individual rectangular plates which make up the element. The plates which make up the element are connected along the mating lines, therefore when looking at the stability of specific plates the influence of adjacent plates cannot be overlooked. Adjacent plates when loaded axially in this configuration act like elastic clamping, they load the given plate with bending moments evenly distributed along the mating lines. These moments are [1, 2] proportional to the plate bend on those edges, and the coefficient of proportionality C – stiffness of elastic clamping – depend on the adjacent plates to which the plate is connected. These cases occur when observing stability of any thin-walled beam with a rectangular cross-section contour, as well as when considering rib stability of thin-walled beams with a U and Z cross-section. Buckling calculations of beam elements is an indispensable part of calculating all axially loaded structures [3-6].

1. ANALYSIS OF A PLATE ELASTICALLY CLAMPED ALONG THE EDGES

The problem considered in this research is buckling of beams which is based on the analysis of the buckling of a rectangular elastically clamped plate whose length is equal to the length of the beam, and the width b is equal to the length of the contour line of the profile (Figure 1) [7]. The plate is elastically clamped along the longitudinal edges.

The stiffness of elastically clamped longitudinal edges are different on either side (there is no symmetry of the system, it is considered to be a general case) and continuous along the edge of the plate. Along the other two loaded edges the plate is freely placed.



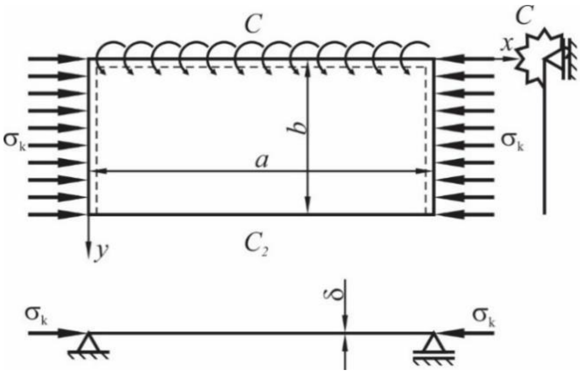


Figure 1 Plate supports

Buckling of the plate in the longitudinal direction, due to the way it is supported, can be presented as a sinusoid using the following expression:

$$f(x)=\sin\frac{m\pi x}{a} \tag{1}$$

Plate buckling in the transverse direction is determined by the use of superposition of the free support, when the buckling can represent a sinusoidal function, and the buckling of two "elastic" moments:

$$f(y)=\sin\frac{\pi y}{b}-N_1\left(2b^2y-by^2+y^3\right)-N_2\left(b^2y-y^3\right) \tag{2}$$

Deflection of the buckled plate can be represented using the following expression:

$$w=Af(x)f(y)=A\sin\frac{m\pi x}{a}\left[\sin\frac{\pi y}{b}-N_1\left(2b^2y-by^2+y^3\right)-N_2\left(b^2y-y^3\right)\right] \tag{3}$$

where: *A* – constant (buckling amplitude); *x, y* – longitudinal and transversal coordinate; *m* – number of longitudinal half-waves of the deformed plate ***(m*** $\in \mathbb{N}$***); N₁*** and $\mathbf{N_2}$ – constants which are determined from limit conditions.

Bending moments of the plate on the edges must be equal to the moments of the elastic clamping:

$$\begin{aligned}-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=0}&=C_1\left(\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=0},\\ -D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=b}&=-C_2\left(\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=b}.\end{aligned} \tag{4}$$

where $D=\frac{E\delta^3}{12\left(1-\mu^2\right)}$ – plate bending stiffness (E – Young modulus, μ - Poisson coefficient).

From these conditions (4) constants N_1 and N_2 are derived to have the following values:

$$\begin{aligned}-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=0}&=C_1\left(\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=0},\\ -D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=b}&=-C_2\left(\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=b}.\end{aligned} \tag{5}$$

that is:

$$N_1 = \frac{m^2\pi^3\overline{C_1}\left(6+m^2\pi^2\overline{C_2}\right)}{3b^3\left(12+4m^2\pi^2\overline{C_1}+4m^2\pi^2\overline{C_2}+m^4\pi^4\overline{C_1}\,\overline{C_2}\right)},$$
$$N_2 = \frac{m^2\pi^3\overline{C_2}\left(6+m^2\pi^2\overline{C_1}\right)}{3b^3\left(12+4m^2\pi^2\overline{C_1}+4m^2\pi^2\overline{C_2}+m^4\pi^4\overline{C_1}\,\overline{C_2}\right)}. \tag{6}$$

where $\overline{C_1} = \frac{bC_1}{a^2D}$ and $\overline{C_2} = \frac{bC_2}{a^2D}$ - is reduction (non-dimensional) of the elastic clamping stiffness. With joint supports $C_1 = C_2 = 0$, $\overline{C_1} = \overline{C_2} = 0$, and $N_1 = N_2 = 0$, and with clamping it is $C_1 = C_2 = \infty$, $\overline{C_1} = \overline{C_2} = \infty$, $N_1 = \frac{\pi}{3b^3}$ and $N_2 = \frac{\pi}{3b^3}$. In the general case $0 \le N_1 \le \frac{\pi}{3b^3}$ and $0 \le N_2 \le \frac{\pi}{3b^3}$.

Potential energy of the system van be determined as a sum of potential energies of the plate and the potential energy of the elastic clamps:

$$E_p = \frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}dxdy + \int_0^a\left(\frac{M_1^2}{2C_1}\right)dx + \int_0^a\left(\frac{M_2^2}{2C_2}\right)dx \tag{7}$$

In this case buckling was considered using function (3) and that moments are $M_1 = C_1\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{y=0}$ and $M_2 = -C_2\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{y=a}$.

Work of external forces is:

$$A_d = \frac{1}{2}\sigma_k\delta\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{8}$$

By equating the potential energies in the system (7) and the work of external forces (8) critical load can be given as:

$$\sigma_k = \frac{D\pi^2}{b^2\delta}k \tag{9}$$

where k is the buckling coefficient:

$$k = m^2\left(\frac{b}{a}\right)^2 + \frac{1}{m^2}\left(\frac{a}{b}\right)^2 Q + R \tag{10}$$

and P and Q are dimensionless constants:

$$P = 105\frac{\left[12+4m^2\pi^2\left(\overline{C_1}+\overline{C_2}\right)+m^4\pi^4\overline{C_1}\,\overline{C_2}\right]\left[12+\left(4m^2\pi^2-24m^2\right)\left(\overline{C_1}+\overline{C_2}\right)+\left(m^4\pi^4-8m^4\pi^2\right)\overline{C_1}\,\overline{C_2}\right]}{\begin{gathered}15120+10080m^2\left(\pi^2-6\right)\left(\overline{C_1}+\overline{C_2}\right)+16m^4\pi^2\left(105\pi^2+4\pi^4-1260\right)\left(\overline{C_1}^2+\overline{C_2}^2\right)+\\ +4m^4\pi^2\left(1470\pi^2+31\pi^4-15120\right)\overline{C_1}\,\overline{C_2}+\\ +42m^6\pi^4\left(20\pi^2+\pi^4-280\right)\left(\overline{C_1}^2\overline{C_2}+\overline{C_1}\,\overline{C_2}^2\right)+7m^8\pi^6\left(15\pi^2+\pi^4-240\right)\overline{C_1}^2\overline{C_2}^2\end{gathered}}, \tag{11}$$

$$Q = 14\frac{2160 + 1440m^2\left(\pi^2-6\right)\left(\overline{C}_1+\overline{C}_2\right) + 48m^4\pi^2\left(7\pi^2-60\right)\left(\overline{C}_1^{\,2}+\overline{C}_2^{\,2}\right) + 144m^4\pi^2\left(7\pi^2-60\right)\overline{C}_1\overline{C}_2 + 60m^6\pi^4\left(3\pi^2-28\right)\left(\overline{C}_1^{\,2}\overline{C}_2+\overline{C}_1\overline{C}_2^{\,2}\right) + 5m^8\pi^6\left(5\pi^2-48\right)\left(\overline{C}_1^{\,2}\overline{C}_2^{\,2}\right)}{15120 + 10080m^2\left(\pi^2-6\right)\left(\overline{C}_1+\overline{C}_2\right) + 16m^4\pi^2\left(105\pi^2+4\pi^4-1260\right)\left(\overline{C}_1^{\,2}+\overline{C}_2^{\,2}\right) + 4m^4\pi^2\left(1470\pi^2+31\pi^4-15120\right)\overline{C}_1\overline{C}_2 + 42m^6\pi^4\left(20\pi^2+\pi^4-280\right)\left(\overline{C}_1^{\,2}\overline{C}_2+\overline{C}_1\overline{C}_2^{\,2}\right) + 7m^8\pi^6\left(15\pi^2+\pi^4-240\right)\overline{C}_1^{\,2}\overline{C}_2^{\,2}}. \tag{12}$$

Buckling coefficients for some extreme cases of supports (joints along all for edges, clamping along one plate edge and clamping on other edges, as well as two sided clamping) are the same as values given in [6]. For the case of elastic supports the coefficient of buckling is determined by the expressions given in Figure 2.

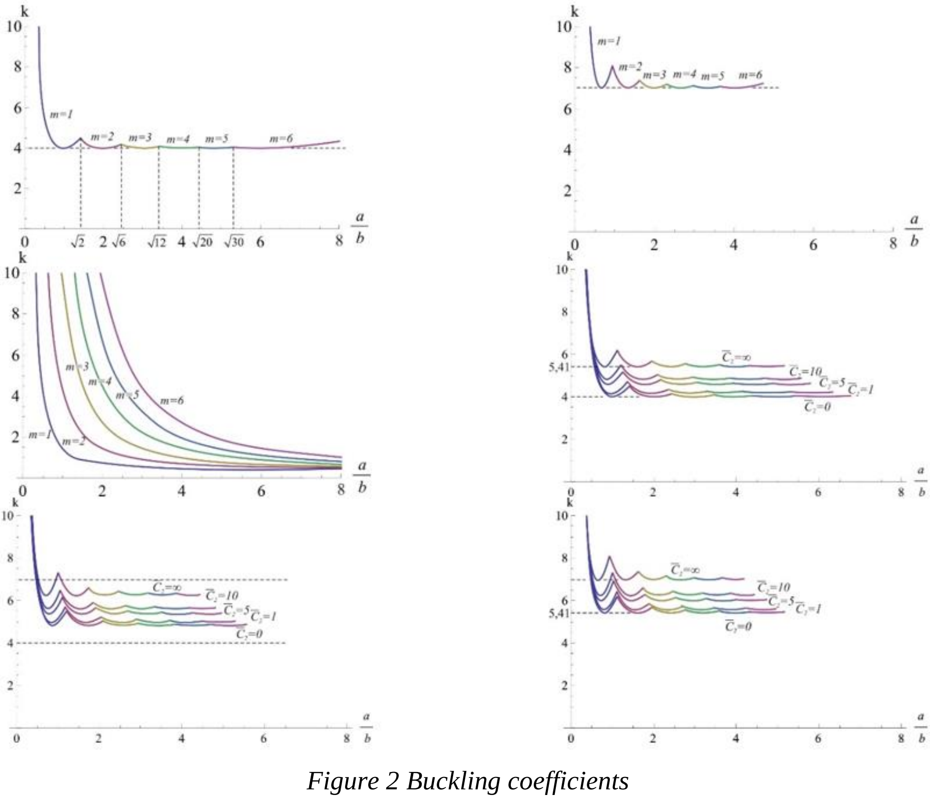


Figure 2 Buckling coefficients

2. TORSION STIFFNESS

In order to calculate the local buckling stress in general cases of elastically clamped plates and to use them in analysis of the plate's stability the torsion stiffness must be determined for buckling of prismatic shells. It is important to note that prismatic shell loses stability as a whole the instant when the weakest plate loses stability, that is to say that if the shell consists of ribs and an outer shell, the shell loses its stability when any one of the ribs or the outer shell loose stability. Figure 3 shows two general forms of plate with elastic clamping, as well as the shape used for each case of prismatic shell [6, 7]. One form is the plate freely supported along the edges which are under load and elastically supported along the longitudinal edges, and the other is a plate freely supported along the edges which are under load, elastically supported along one and free along the other longitudinal edge. Equations for determining critical buckling stress for these two support cases are determined in the previous heading of this paper [8, 9]. When analyzing plates which are integral parts of the shell, regardless of the load-case it is necessary to introduce a correctional factor [9, 10]:

$$r = 1 - \frac{\sigma_{restriktied\ plate}}{\sigma_{restraining\ plate}} \tag{13}$$

In the case of buckling equation (12) is given as:

$$r = 1 - \frac{\sigma_k^{restricted\ plate}}{\sigma_k^{restraining\ plate}} \quad x = a^2 + b_1 + \sum \varepsilon_i \alpha \tag{14}$$

It is important to note that when calculating correctional factors for critical stress of limited and limiting plates the critical stress of the plate with the free support, that is the freely supported plate and plate whose one edge is free and the other is freely supported.

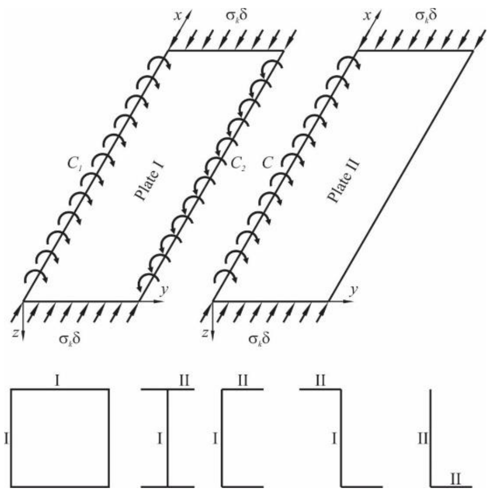


Figure 3 Elastically clamped plates

Torsional stiffness for rectangular profiles is:

$$C = \frac{2m\pi D\left[\cosh\left(\frac{b_r m\pi}{a}\right) - 1\right]}{a\sinh\left(\frac{b_r m\pi}{a}\right) - b_r m\pi}\left(1 - \frac{b_r^4}{b_p^4}\frac{\left(a^2 + b_p^2\right)^2}{\left(a^2 + b_r^2\right)^2}\right) \quad x = a^2 + b_1 + \sum \varepsilon_i \alpha \tag{15}$$

While when using I profiles:

$$C = \frac{2m\pi D\left[a\sinh\left(\frac{2b_p m\pi}{a}\right) - 2b_p m\pi\right]}{7a^2 - 2b_p^2 m^2\pi^2 + a^2\cosh\left(\frac{2b_p m\pi}{a}\right)}\left\{1 - \frac{b_p^2\left(a^2 + b_r^2 m^2\right)^2 \pi^2}{b_r^4 m^2\left[b_p^2 m^2\pi^2 + 6a^2(1-\mu)\right]}\right\} \tag{16}$$

When using Z and U profiles

$$C = \frac{2m\pi D\left[a\sinh\left(\frac{2b_p m\pi}{a}\right) - 2b_p m\pi\right]}{7a^2 - 2b_p^2 m^2\pi^2 + a^2\cosh\left(\frac{2b_p m\pi}{a}\right)}\left\{1 - \frac{b_p^2\left(a^2 + b_r^2 m^2\right)^2 \pi^2}{b_r^4 m^2\left[b_p^2 m^2\pi^2 + 6a^2(1-\mu)\right]}\right\} \tag{17}$$

3. NUMERICAL CALCULATION

For validation of the results calculated in the previous heading available data from literature was used (analytical and experimental data) and numerical analysis using the finite element method (FEM). Pre-processing and post-processing in the FEM analysis was done in FEMAP software, and processing was done in NXNastran which is integrated in FEMAP. For solving this problem the default input settings were used.

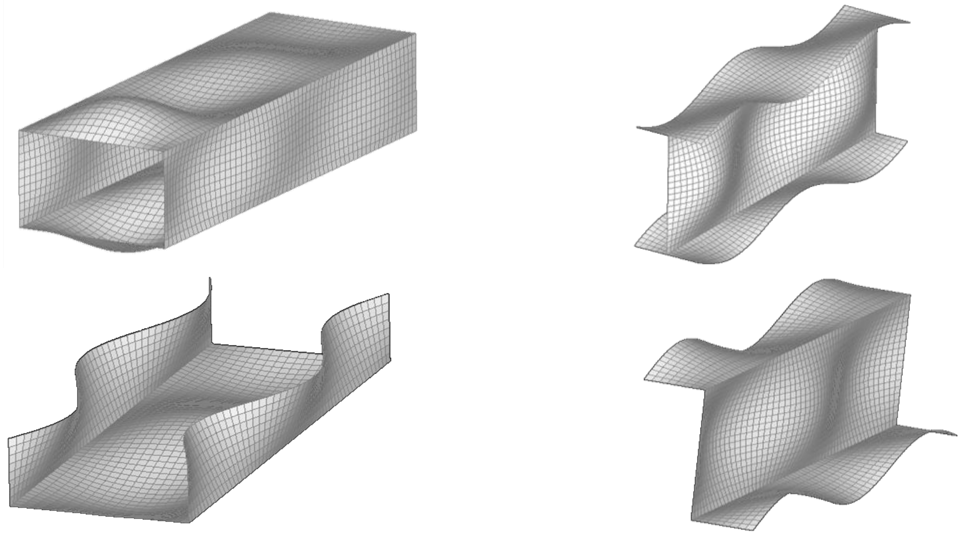


Figure 4 FEA results

The greatest variation recorded is 8.55%, therefore the analytical calculation equations can be considered acceptable for stability calculation. Verification of data with literature also confirms coincidence with the results from the literature are in the range of 5.5% [6, 7] and [11-13].

4. CONCLUSIONS

The use of equations for determining torsional rigidity give equations for determining critical buckling stress of various thin-walled profiles which represent prismatic shells. By analysing the equations for determining critical stress of rectangular profiles it can be

concluded that rectangular profiles will always buckle along the wider side. In the case of square profiles there is no influence of elastic supports however the plates which make up the profile act as freely supported plates. By decreasing the width of the narrow sides the influence on the stability of the wider sides increases, and with it the influence on the shell stability. In the limiting cases when the width of the narrower sides is converging to zero the elastic supports act as regular supports. By analysing the equations for determining critical loads for U and Z profiles the following conclusions can be made. Thin walled beams which use U and Z profiles can be found in two cases depending on the proportions of the outer shell and ribs. The first case is when the outer shell buckles first and when the rib profile represents the limiting element. This is the case when the width ratio of outer shell to rib is greater than 0.36847. As the width ratio increases over this value the influence of the rib is lessened. In the case that the width ratio is equal to this number there is no mutual influence of the rib and outer shell and they function as if they are freely supported plates. The other case is when the rib buckles first and the limiting element is the outer shell. This is the case when the width ratio of outer shell to rib is less than 0.36847. By decreasing this ratio the system stiffness increases up to the point when the ratio reaches 0.2. Further decreasing of the ratio leads to a decrease in rib stiffness and when the rib width converges to zero the influence is completely lost. These results are used in data taken from literature.
Analysing I profiles the same conclusions can be made as with the U and Z profiles, except in this case half the belt width is used. When analysing L profiles the first to buckle is always the wider rib. When the ribs are equal widths there is no influence of the elastic support. Torsional stiffness is negligible (the maximal value of reduced torsional stiffness is 0.7795). Due to this it is understandable why in literature only uneven L profiles can be found.

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